NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

Library

NO. 1957122

IMM-NYU 243
DECEMBER 1957

NEW YORK UNIVERSITY
INSTITUTE OF
MATHEMATICAL SCIENCES

Note on Assignment Problems

MARTIN KNESER

PREPARED UNDER CONTRACT NO. Nonr-285(32)
WITH THE OFFICE OF NAVAL RESEARCH,
UNITED STATES NAVY
AND CONTRACT NO. NSF-G4615 WITH THE
NATIONAL SCIENCE FOUNDATION

REPRODUCTION IN WHOLE OR IN PART
IS PERMITTED FOR ANY PURPOSE
OF THE UNITED STATES GOVERNMENT.

IMM-243
c.1

No. 1957122

IMM-NYU 243
December 1957

New York University
Institute of Mathematical Sciences

NOTE ON ASSIGNMENT PROBLEMS

Martin Kneser

Prepared under the auspices of Contract
Nonr-285(32) with the Office of Naval
Research, United States Navy and
Contract NSF-G4615 with the National
Science Foundation.

New York, 1957

Note on Assignment Problems

Martin Kneser

1. The problem. Let $A = (a_{ik})$ be an $n \times n$ matrix with nonnegative elements. The problem of finding a permutation s of the numbers $1,2,\ldots,n$ which minimizes the sum $A(s) = \sum_{i=1}^{n} a_{i,s(i)}$ is the mathematical content of the assignment problem [1]. A method for the practical solution of this problem for large n was given by Flood and others [2]. This method is considerably simpler than the (practically impossible) determination of all $n!$ values $A(s)$. It depends on the following two remarks.

(1) The minimal permutations s of the matrix $B = (a_{ik}-u_i-v_k)$ with any numbers u_i, v_k are the same as for A, because $B(s) = A(s)-c$, with $c = \sum_{i=1}^{n} u_i + \sum_{k=1}^{n} v_k$ independent of s.

(2) If $a_{i,s(i)} = 0$ for $i = 1,2,\ldots,n$, then s is a minimal permutation.

The method consists in a transformation of A by (1) into a matrix to which (2) applies. As a preliminary step, we may apply (1) with $u_i = \min_k a_{ik}$, $v_k = 0$. The result is a matrix $B = (b_{ik})$ with $\min_k b_{ik} = 0$, $i = 1,2,\ldots,n$. By a second application of (1) with rows and columns interchanged, we get a matrix $C = (c_{ik})$ with $\min_k c_{ik} = \min_i c_{ik} = 0$, i.e. every row and every column of C contains at least one zero.

Consider now the set of zeros in the matrix C, and let r be the minimum number of lines (by 'line' we mean any row or column) required to cover all these zeros. Let $i \in I = \langle i_1,\ldots,i_s \rangle$

respectively. $k \in K = \langle k_1, \ldots, k_{r-s} \rangle$ be the indices of the rows respectively columns of such a covering. Application of (1) with

$$u_i = -v_k = \min_{j \notin I, \ell \notin K} c_{j\ell} = c \qquad \text{if} \quad i \in I \,, \quad k \notin K \,,$$

$$u_i = \ v_k = 0 \qquad \text{if} \quad i \notin I \,, \quad k \in K \,,$$

gives a new matrix $D = (d_{ik})$ with

$$d_{ik} = \begin{cases} c_{ik} - c & \text{if } i \notin I,\ k \notin K, \\ c_{ik} & \text{if } I \in I,\ k \notin K \text{ or } i \notin I,\ k \in K\,, \\ c_{ik} + c & \text{if } i \in I,\ k \in K\,. \end{cases}$$

To this matrix, we may apply the same process, etc. If a matrix with $r = n$ is reached this way, a theorem of König [3] asserts the existence of a permutation s wuth property (2). The question is, whether this necessarily happens. Flood [2] did not prove this, nor give an estimate for the number N of steps required. The following example of a 5×5 matrix shows that N may be arbitrarily large for fixed n.

Let a,b be two positive integers, and

$$C_{a,b} = \begin{pmatrix} 2a & a & a & 0 & 0 \\ a & 0 & 0 & b & b \\ a & 0 & 1 & b & b \\ 0 & b & b & 2b & 2b \\ 0 & b & b & 2b & 2b \end{pmatrix} .$$

The preliminary step is unnecessary, since every row and every column contains a zero. Our process with $I = K = \langle 1,2\rangle$ gives $c = 1$ and

$$D_{a,b} = \begin{pmatrix} 2a+1 & a+1 & a & 0 & 0 \\ a+1 & 1 & 0 & b & b \\ a & 0 & 0 & b-1 & b-1 \\ 0 & b & b-1 & 2b-1 & 2b-1 \\ 0 & b & b-1 & 2b-1 & 2b-1 \end{pmatrix}$$

The next step with $I = K = \langle 1,3\rangle$ carries $D_{a,b}$ over into

$$\begin{pmatrix} 2a+2 & a+1 & a+1 & 0 & 0 \\ a+1 & 0 & 0 & b+1 & b+1 \\ a+1 & 0 & 1 & b+1 & b+1 \\ 0 & b+1 & b+1 & 2b+2 & 2b+2 \\ 0 & b+1 & b+1 & 2b+2 & 2b+2 \end{pmatrix}$$

which is exactly $C_{a+1,b-1}$. Two further steps lead to $C_{a+2,b-2}$, etc., until after $N = 2b$ steps we get $C_{a+b,0}$, to which (2) applies. Since b is arbitrary, the number of steps may be arbitrarily large. The reason for this is that there are several possibilities for covering the zeros of $C_{a,b}$ and $D_{a,b}$ with four lines, and we made a bad choice. It is the purpose of the next two paragraphs to show that with a proper choice of these coverings the number of steps is always finite and at most $\frac{1}{2}(n-1)(n-2)$ for an $n \times n$ matrix.

2. Estimation of the number of steps. In order to avoid the possibility of the given example, we make the following assumption. If several coverings of the zeros in the matrix C with the minimal number $r = r(C)$ of lines are possible, we choose one with the least possible number $s = s(C)$ of rows. Our estimation now depends on the fact that either

$$r(D) > r(C) \tag{4a}$$

or

$$r(D) = r(C) \quad \text{and} \quad s(D) > s(C) \tag{4b}$$

holds. Let us assume this for a moment. Then, since C and consequently D and all the other derived matrices have at least one zero in every row and every column, the value of r is always ≥ 2, and s cannot be 0 or r, as long as $r < n$. So the worst possible case corresponds to a series of matrices C,D,... with values (r,s) equal to $(2,1)$, $(3,1)$, $(3,2)$, $(4,1),\ldots,(n-1,1)$, $(n-1,2),\ldots,(n-1,n-2)$, $(n,1)$, the last matrix having property (2) for some permutation s by Königs theorem [3]. This gives the desired estimate $\frac{1}{2}(n-1)(n-2)$ for the number of steps.

3. Proof of (4). We distinguish three classes of zeros in the matrix C. First the $c_{ik} = 0$ with $i \in I$, $k \notin K$; second, those with $i \notin I$, $k \in K$ and third, those with $i \in I$, $k \in K$. As all zeros are covered by the rows with index $i \in I$ and the columns with index $k \quad K$, there are no further zeros in C. By (3), the zeros of the first and second class are zeros of D too.

But there is at least one further element $d_{ik} = 0$ in D with $i \notin I$, $k \notin K$, namely for a pair (i,k) where the minimum $c = \min_{i \in I, k \notin K} c_{ik}$ occurs. In a minimal covering of the zeros of D, we have two subclasses of lines: those which contain a zero of the first or second class respectively. These classes have no line in common, because no zero of the first class is on a line with one of the second class. The lines of the first class, together with the r(C)-s(C) columns with index $k \in K$ cover all zeros of C. So there are at least s(C) lines in the first class, by definition of r(C), and if there are exactly s(C), these must all be rows, by definition of s(C). Similarly, there are at least r(C)-s(C) lines in the second class. This proves $r(D) \geq r(C)$, and if $r(D) = r(C)$, then $s(D) \geq s(C)$. It remains to exclude the case $r(D) = r(C)$, $s(D) = s(C)$. Here $r(D) = r(C)$ implies that every line of the covering is in one of the two classes, and that the first class consists of s(C) rows. As a consequence of $s(D) = s(C)$, the r(D)-s(D) lines of the second class must all be columns. But every row of the first class has an index $i \in I$, and every column of the second class an index $k \in K$, and so cannot cover the additional zero d_{ik} with $i \notin I$, $k \notin K$. This is a contradiction, and so our proof is finished.

Bibliography

[1] Introduction to Operations Research, John Wiley and Sons, 1957, Chap. 12.

[2] M.M. Flood, The Traveling Salesman Problem, J. Opns. Res. Soc. Am., 4, No. 1, 61-75, (1956).

[3] D. König, Theorie der endlichen und unendlichen Graphen, p. 240; for a very simple proof of the equivalent "Marriage problem" see P. Halmos and H. Vaugham, Amer. J. Math., 72, (1950) pp. 214-215 or W. Maak, Fastperiodische Funktionen, p. 234.

Distribution List

Armed Services Technical Information Agency
Document Service Center
Knott Building
Dayton 2, Ohio (5)

Commanding Officer
Office of Naval Research Branch Office
The John Crerar Library Building
86 East Randolph Street
Chicago 1, Illinois (1)

Commanding Officer
Office of Naval Research Branch Office
346 Broadway
New York 13, New York (1)

Commanding Officer
Office of Naval Research Branch Office
Navy No. 100, c/o Fleet Post Office
New York, New York (2)

Commanding Officer
Office of Naval Research Branch Office
1030 E. Green Street
Pasadean 1, California (1)

Logistics Branch
Room 2718, Code 436
Office of Naval Research
Washington 25, D. C. (2)

Industrial College of the Armed Forces
Fort Lesley J. McNair
Washington 25, D. C.
Attn: Mr. L. L. Henkel (1)

Director
National Science Foundation
Washington 25, D. C. (1)

Director
National Security Agency
Washington 25, D. C. (1)

Directory
Naval Research Laboratory
Washington 25, D. C.
Attn: Technical Information Officer (6)

Naval War College
Logistics Department
Luce Hall
Newport, Rhode Island (1)

Office of Technical Services
Department of Commerce
Washington 25, D. C. (1)

Professor Kenneth J. Arrow
Department of Economics
Stanford University
Stanford, California (1)

Dean L.M.K. Boelter
Department of Engineering
University of California
Los Angeles 24, California (1)

Dr, A. Charnes
School of Industrial Engineering and Management
Purdue University
Lafayette, Indiana (1)

Professor William W. Cooper
Graduate School of Industrial Administration
Carnegie Institute of Technology
Pittsburgh 13, Pennsylvania (1)

Rear Admiral H.E. Eccles, USN, (RET)
101 Washington Street
Newport, Rhode Island

Professor David Gale
Department of Mathematics
Brown University
Providence, Rhode Island (1)

Mr. Murray A. Geisler
The RAND Corporation
1700 Main Street
Santa Monica, California (1)

Dr. J. Heller
Navy Management Office
Washington 25, D. C. (1)

Dr. C. C. Holt
Graduate School of Industrial Engineering
Carnegie Institute of Technology
Schenley Park
Pittsburgh 13, Pennsylvania (1)

Professor J. R. Jackson
Management Sciences Research Project
University of California
Los Angeles 24, California (1)

Professor Samuel Karlin
Department of Mathematics
Stanford University
Stanford, California (1)

Professor S. B. Littauer
Department of Industrial Engineering
Columbia University
409 Engineering Building
New York 27, New York (1)

Dr. W. H. Marlow
The George Washington University
Logistics Research Project
707 22nd Street
Washington 7, D. C. (1)

Professor Jacob Marschak
Cowles Commission
Yale University
New Haven, Connecticut (1)

Professor Oskar Morgenstern
Economics Research Project
Princeton University
9-11 Lower Pyne Building
92 A Nassau Street
Princeton, New Jersey (1)

Professor R. R. O'Neill
Department of Engineering
University of California
Los Angeles, California (1)

Professor H. N. Shapiro
New York University
Institute of Mathematical Sciences
New York, New York (1)

Professor H. A. Simon
Head, Department of Industrial Administration
Carnegie Institute of Technology
Schenley Park
Pittsburgh 13, Pennsylvania (1)

Professor R. M. Thrall
Department of Mathematics
University of Michigan
Ann Arbor, Michigan (1)

Professor L. M. Tichvinsky
Department of Engineering
University of California
Berkeley 4, California (1)

Professor James Tobin
Cowles Foundation for Research in Economics
Box 2125, Yale Station
New Haven, Connecticut (1)

Dr. C. B. Tompkins, Director
Numerical Analysis Research
University of California
405 Hilgard Avenue
Los Angeles 24, California (1)

Professor A. W. Tucker
Department of Mathematics
Fine Hall, Box 708
Princeton University
Princeton, New Jersey (1)

Professor Jacob Wolfowitz
Department of Mathematics
Cornell University
Ithaca, New York (1)

Dr. Edward Barankin
University of California
Berkeley, California (1)

Dr. Richard Bellman
The RAND Corporation
1700 Main Street
Santa Monica, California (1)

Professor S. S. Cairns, Head
Department of Mathematics
University of Illinois
Urbana, Illinois (1)

Dr. E. W. Cannon
Applied Mathematics Division
National Bureau of Standards
Washington 25, D. C. (1)

Dr. Randolph Church, Chairman
Department of Mathematics and
Mechanics
U.S. Naval Postgraduate School
Monterey, California (1)

Dr. George B. Dantzig
The RAND Corporation
1700 Main Street
Santa Monica, California (1)

Dr. Walter Jacobs
Hdqs. USAF, DCS/Comptroller
AFADA-3D, The Pentagon
Washington 25, D. C. (1)

CAPT W. H. Keen
Staff, Commander Fleet Air Wing
Western Pacific
c/o Fleet Post Office
San Francisco, California (1)

Professor Harold Kuhn
Department of Mathematics
Bryn Mawr College
Bryn Mawr 6, Pennsylvania (1)

Dr. Max A. Woodbury
Department of Mathematics
New York University
University Heights
New York 53, New York (1)

Dr. Richard A. Miller
4071 West 7th Street
Fort Worth 7, Texas (1)

Director
Operations Evaluating Group
Office of Chief of Naval
Operations (OP 03EG)
Navy Department
Washington 25, D. C. (1)

Captain W. H. Keen
Aircraft Design Division
Bureau of Aeronautics
Navy Department
Washington 25, D. C. (1)

Superintendent
U.S. Naval Postgraduate School
Monterey, California
Attn: Library (1)

Electronic Computer Division
(Code (280)
Bureau of Ships
Department of the Navy
Washington 25, D. C. (1)

C. E. Lemke
Mason House
Department of Mathematics
Rensselaer Polytechnical Institute
Troy, New York

Alan J. Hoffman
General Electric Company
Management Consultation Services
570 Lexington Avenue
New York 22, New York

Date Due

MAY 20 '98

PRINTED IN U. S. A.

NYU
IMM-
243 Kneser.
Note on assignment problems.

NYU
IMM-
243 Kneser.
AUTHOR
Note on assignment
TITLE
problems.

DATE DUE	BORROWER'S NAME	ROOM NUMBER
AY 20 '65	~~Paul Mazin~~	

N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.